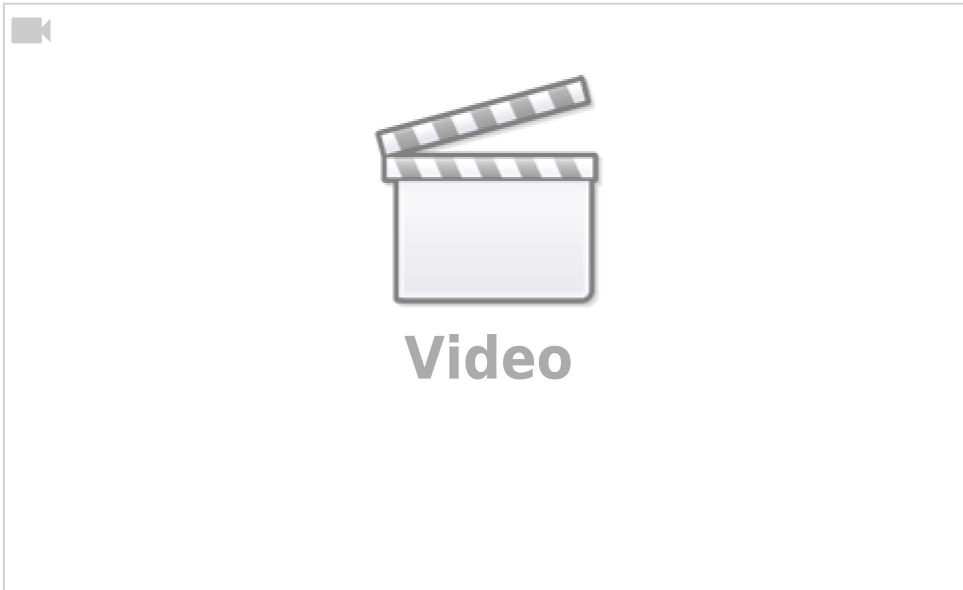
Around or Through

Here's a cool physics puzzle I saw in 2015:



I thought about it for a while, thinking I'd be able to reason my way through it on intuition alone, but I couldn't. So here's the math!

Update: four years later, I found a semi-intuitive argument that I think works, but it still requires some math….

Preliminary Work

Let F_g be that the force of gravity between the planet of mass M and the particle of mass m at a distance r from the centre. The force of gravity between the particle and the planet is then:

$ F_g = - \dfrac{GMm}{r^2} $, where G is the gravitational constant.

If the planet has a uniform density ρ, then its mass is:

$$ M = \rho V = \frac{4}{3} \rho \pi r^3 $$

So at the surface or inside (Gauss' law thing), the force is:

$$ F_g = - \frac{4}{3} \pi G \rho m r $$

In Orbit

For a particle in orbit at the surface, the centripetal (or is it centrifugal?) force must balance the gravitational force so:

$$ \frac{4}{3} \pi G \rho m r = \frac{m v^2}{r} $$

But the velocity v is the distance it takes to get to the other side (which is half of the circumference) over the time it takes (which is what we're really after):

$$ v = \frac{\pi r}{t} $$

$$ \Rightarrow \frac{4}{3} \pi G \rho m r = \frac{m \pi^2 r^2}{rt^2} $$

$$ \Rightarrow t = \sqrt{\frac{3 \pi}{4 G \rho}} $$

Cool, so the time doesn't actually depend on the size of the planet (there's no r in the equation!) To get a rough idea of what that would be for Earth, I just looked up the average density ($\rho = 5513 \text{ kg/m}^3$):

$$ \Rightarrow t = \sqrt{\frac{3 \pi}{4 \times 6.673 \times 10^{-11} \times 5513}} \approx 2531 \text{ s} \approx 42 \text{ min } 11 \text{ sec} $$

Wikipedia says that the orbital period of the International Space Station is about 93 minutes so half of that is 46 min 21 sec, which is pretty darn close!

Falling Through

While falling through, the particle experiences the force of gravity only from the mass “below” it (Gauss' law thing), so M is itself a function of r.

Which means that the force on the particle is:

$$ F_g(r) = - \frac{4}{3} \pi G \rho m r $$

It's the same force as what we had before, but this time, r is not constant. In fact, you can kind of see it as: mgh but with $g = \frac{4}{3} \pi G \rho$

To find the equation of motion, we can use $F = ma$:

$$ma = - \frac{4}{3} \pi G \rho m r$$

$$\Rightarrow \ddot{r}(t) = - \frac{4}{3} \pi G \rho r$$

$$\Rightarrow r(t) = r_0 \cos\left(\sqrt{\frac{4}{3} \pi G \rho}t\right)$$

I skipped a few steps here, but differentiate $r(t)$ twice and you'll see that $\ddot{r}(t) = -\frac{4}{3} \pi G \rho r$ so it works. Also, I was a bit lazy and changed notation midway: $r(t)$ is now the position of the particle as a function of time, and r_0 is the radius of the planet.

But all this doesn't really matter, the important part is that if left alone, the particle would oscillate back and forth at an angular frequency of $\sqrt{\frac{4}{3} \pi G \rho}$. Which means that to cover half of the period of oscillation (ie, get to the other side of the planet):

$$ \sqrt{\frac{4}{3} \pi G \rho} t = \pi $$

$$ t = \sqrt{\frac{3 \pi}{4 G \rho}} $$

Which is the same as the going around in orbit!!!

A Semi Intuitive Argument

If we go back to the first section, we had that the force at the surface or inside (Gauss' law thing) is:

$$ F_g (r) = - \frac{4}{3} \pi G \rho m r $$

In the case of the particle in orbit, the force always has the same magnitude but its direction keeps changing (always pointing toward the centre). Another way of looking at this is to break up the force into its vertical and horizontal components. If we look only at the vertical component, the force would be the force at the surface scaled by sine of the angle:

$$F(r_0) \sin\theta$$

Note also that since the particle is already in orbit, the initial vertical component of the speed at the North pole is zero because it's travelling only the horizontal direction (at the North pole).

Now, the particle dropped through the planet is acted on by (presumably) a different force since it varies with r. But we can also write that variable as a function of the radius of the planet and the angle: $r = r_0 \sin \theta$. This gives us the same force as the particle in orbit: $F(r_0) \sin\theta$

Since the (vertical) forces on the particles is the same, and since they both have the same initial (vertical) velocities, they must follow the exact same vertical path at the same time.

This line of reasoning shows that it takes the same time for them to get to the other side, but it doesn't tell us what that time is. It also requires doing the preliminary work to find that the force is a linear function of r, and to understand the use of Gauss' law inside the planet. It's not a purely intuitive solution, but it's as close as I could get.

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